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Decidable (= separable) objects and morphisms in lextensive categories[1]

A. Carboni [a,*], G. Janelidze [b]

[a] Dipartimento di Matematica, via L.B. Alberti 4, 16132 Genova, Italy

[b] Mathematics Institute of the Georgian Academy of Sciences, Tbilisi 380093, Georgia

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Abstract

After investigating all conceivable properties of decidable objects and maps in left exact categories with well-behaved finite sums (*'lextensive categories'*), we give a *characterization* in such categories of decidable morphisms which are *(finite) coverings* (in an appropriate sense). Finally, we give two applications of this result, to separable algebras and to local homeomorphisms. In both cases it explains categorically the advantage of two well-known notions – *strongly* separable algebras and local homeomorphisms with *path lifting property*, respectively.

Keywords: Extensive categories; Separable algebras; Coverings; Path lifting property

1. Introduction: lextensive categories

A *lextensive category* is a lex category $\mathbb{C}$ with finite sums and which is *extensive*: for each pair of objects X, Y, the canonical functor between comma categories

$$(\mathbb{C}\downarrow X) \times (\mathbb{C}\downarrow Y) \xrightarrow{+} (\mathbb{C}\downarrow X + Y)$$

is an *equivalence*.

This elegant formulation due to Schanuel (see [16]) turns out to be equivalent to the well-known property of the sums being *disjoint and universal*. However, since the extensivity axiom is stated only in terms of the sum, his virtue is that we can look at weakenings of the requirement of being lex, weakenings which occur in 'nature', as analyzed in [5]. Observe that mere extensivity implies that the initial object is *strict*,

* Corresponding author.

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that coproduct injections are monomorphisms, that pullback along injections exists, and that the adjoint inverse to the sum functor is given by pulling back along injections.

Another meaningful characterization is in terms of *distributivity*: a category $\mathbb{C}$ with finite products and finite sums is *distributive* when the canonical maps

$$X \times Z + Y \times Z \longrightarrow (X + Y) \times Z$$

are invertible. Now, when $\mathbb{C}$ is lex, then $\mathbb{C}$ is lextensive if and only if sums are *disjoint* and $\mathbb{C}$ is *locally distributive*, i.e. if and only if each comma category $(\mathbb{C} \downarrow U)$ is distributive.

Note that the notion of a lextensive category is stable under slicing, so that if $\mathbb{C}$ is lextensive, then each $(\mathbb{C} \downarrow U)$ is lextensive. A basic construction in the theory of extensive categories is that of the *free* one, meaning the left biadjoint to the forgetful 2-functor

$$\mathrm{EXT} \longrightarrow \mathrm{CAT}$$

from the 2-category EXT of extensive categories to the 2-category CAT of categories. Such biadjoints turn out to be the same as the left biadjoint to the forgetful 2-functor from the 2-category SUM of categories with finite sums to CAT, and is given by the construction of the category Fam($\mathbb{C}$) of the finite families of objects of $\mathbb{C}$ (see e.g. [5]): *objects* of Fam($\mathbb{C}$) are finite families $(C_i)_{i \in I}$ and *arrows*

$$(C_i)_{i \in I} \longrightarrow (C_j)_{j \in J}$$

are pairs given by a function $\phi : I \longrightarrow J$ and a family of maps of $\mathbb{C}$

$$(f_i : C_i \longrightarrow C_{\phi(i)})_{i \in I}.$$

Now the point is that not only Fam($\mathbb{C}$) is the finite coproduct completion of $\mathbb{C}$, but is always extensive; moreover, any kind of limit that the starting category $\mathbb{C}$ may have is preserved by the 'Fam' construction, so that if $\mathbb{C}$ has products, then Fam($\mathbb{C}$) is distributive, and if $\mathbb{C}$ has finite limits, then Fam($\mathbb{C}$) is lextensive. However, we should stress that Fam($\mathbb{C}$) has usually *more* limits than $\mathbb{C}$ has, for instance Fam($\mathbb{C}$) always has pullbacks along coproduct injections, as we already mentioned; in important examples from algebra considered in this paper, Fam($\mathbb{C}$) is lex, hence lextensive, without $\mathbb{C}$ having almost any limit.

Categories of the form Fam($\mathbb{C}$) can be characterized in terms of *connected* objects as follows. Recall that in a category $\mathbb{S}$ with sums, an object C is connected when the representable functor $\mathbb{S}(C, -)$ *preserves finite sums*. Since any decomposition of an object in a finite sum of connected objects is *essentially unique*, categories of the form Fam($\mathbb{C}$) can be characterized as those categories with finite sums for which every object can be written as a finite sum of connected objects. Observe that in an extensive category, an object is connected if and only if it is not initial and it is *indecomposable*, i.e. it cannot be written as a sum of two non-zero objects.

When $\mathbb{C}$ has a terminal object 1, then Fam($\mathbb{C}$) comes equipped with a canonical adjunction

$$\mathrm{Set}_{\mathrm{fin}} \underset{\Delta}{\overset{\pi_0}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{Fam}(\mathbb{C})$$

which is induced by the adjunction

$$1 \underset{1}{\overset{!}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbb{C}$$

and the fact that Fam(1) is the category $\mathrm{Set}_{\mathrm{fin}}$ of finite sets, which is 2-initial in the 2-category EXT* of extensive categories with a terminal object.

Of course, the whole theory can be developed for the infinitary case too, asking all small sums and the infinitary extensive axiom; and it is then clear also how to modify the 'Fam' construction.

Examples of lextensive categories: all toposes are such, but also the category Top of topological spaces, which is infinitary lextensive but is not free as an extensive category, as well as the category Cat of categories, which is infinitary extensive and infinitary free. A related 2-dimensional example is the 2-category of Grothendieck toposes and geometric morphisms between them. Of particular interest are examples of (essentially) algebraic nature: given a lex category $\mathbb{C}$, at most one of the two possibilities can happen: either $\mathbb{C}$-Alg = Lex($\mathbb{C}$,Set) is lextensive, or the dual is, since categories such that both the category and the dual are lextensive are degenerate. Certainly each possibility is a *property* of the theory $\mathbb{C}$, and still a characterization of those $\mathbb{C}$ for which the category of algebras is lextensive or those $\mathbb{C}$ for which the dual of the category of algebras is lextensive is in order (see [6] for some interesting investigations in the case when the dual of the category of algebras is lextensive). The first illuminating example is that of commutative rings: *the dual category of the category $\mathcal{CR}$ commutative rings is in fact lextensive*, as well as the lex theory of commutative rings. Other examples of similar nature, which we will not investigate in the present paper, are given by the dual categories of the category $\mathcal{BA}$ of boolean algebras, and of the category $\mathcal{HA}$ of Heyting algebras. Another interesting example is that of cocommutative coalgebras over a field, which is also free as an (infinitary) extensive category.

The aim of this paper is to investigate a kind of objects and morphisms which in a lextensive category can be defined and have a perfectly good behaviour, as first pointed out by Lawvere in [14], namely the '*decidable* = *separable*' ones: an object is *decidable* or *separable* when the diagonal is a coproduct injection, and a morphism is such when it is so as an object of the comma category. We have to mention two different names for the same notion because there are two important particular cases known under these two different names: *separable algebras* in Galois theory and *decidable objects and morphisms* in topos theory. In the following we will use indifferently both names.

Our first purpose is to show that a large list of basic properties of decidable objects and morphisms, mostly known in the two cases above, can be easily established in general lextensive categories, hence showing the notion of a lextensive category as the appropriate one for the study of these concepts. In particular, we will show that the full subcategory Dec($\mathbb{C}$) of decidable objects of a lextensive category $\mathbb{C}$ is again lextensive and *full on subobjects*, an indication of the nature of decidable objects as a kind of 'discrete' objects. In particular, this implies that if the given lextensive category $\mathbb{C}$ is regular, the Dec($\mathbb{C}$) is also a regular category. It is not clear to us at this writing how to show that our list is in fact the list of *all* properties of decidable objects and morphisms expressible in the language of lextensive categories, which hold in all of them, but we are confident that in fact they are all.

Secondly, in Section 4 we give a *characterization* of separable morphisms which are (*finite*) *coverings* (in an appropriate sense). In Section 5 we give two applications of this result, to separable algebras and to local homeomorphisms. In both cases it explains categorically the advantage of two well-known notions – *strongly* separable algebras and local homeomorphisms with *path lifting property*, respectively.

2. Actions of internal categories

Let $\mathbb{C}$ be a category with pullbacks and let

$$C = \left(C_1 \times_{C_0} C_1 \xrightarrow{m} C_1 \overset{\xrightarrow{d}}{\underset{\xrightarrow{c}}{\xleftarrow{e}}} C_0 \right)$$

be an internal category in $\mathbb{C}$. Recall that an internal C-action on an internal family of objects of $\mathbb{C}$ indexed by the objects of C

$$F = (F_0 \xrightarrow{\pi_F} C_0)$$

is a map

$$C_1 \times_{C_0} F_0 \xrightarrow{\xi_F} C_0$$

from the pullback

$$\begin{array}{ccc} C_1 \times_{C_0} F_0 & \xrightarrow{\pi_2} & F_0 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_F} \\ C_1 & \xrightarrow{d} & C_0 \end{array}$$

such that the following equations hold:

$$\pi_F\,\xi_F = c\,\pi_1, \tag{1}$$

$$\xi_F\,\langle e\,\pi_F, 1_{F_0}\rangle = 1_{F_0}, \tag{2}$$

$$\xi_F\,(m \times_{C_0} 1_{F_0}) = \xi_F\,(1_{C_1} \times_{C_0} \xi_F). \tag{3}$$

A morphism of C-actions $\alpha : F \longrightarrow G$ is a morphism $\alpha : F_0 \longrightarrow G_0$ such that

$$\pi_G\,\alpha = \pi_F \quad \text{and} \quad \xi_G\,(1_{C_1} \times_{C_0} \alpha) = \alpha\,\xi_F. \tag{4}$$

C-actions and morphisms between them form a category which we will denote by $\mathbb{C}^C$. Recall that when $\mathbb{C}$ is the category Set of sets, then $\mathbb{C}^C$ is nothing but the functor category $[\mathbb{C}, \mathrm{Set}]$. If C is an internal *groupoid*, i.e. has an inverse

$$C_1 \xrightarrow{(-)^{-1}} C_1,$$

satisfying

$$d\,(-)^{-1} = c, \tag{5}$$

$$m\,\langle 1_{C_1}, (-)^{-1}\rangle = e\,c, \tag{6}$$

$$m\,\langle (-)^{-1}, 1_{C_1}\rangle = e\,d, \tag{7}$$

then by Eqs. (1) and (5) there exists always the map

$$\xi_F^* = \langle (-)^{-1}\,\pi_1, \xi_F\rangle : C_1 \times_{C_0} F_0 \longrightarrow C_1 \times_{C_0} F_0.$$

Let us recall the following known lemma:

Lemma 1. *When C is an internal groupoid, then*:
(i) $\xi_F\,\xi_F^* = \pi_2 : C_1 \times_{C_0} F_0 \longrightarrow F_0$;
(ii) *if $\alpha : F \longrightarrow G$ is a morphism of actions, then the naturality square*

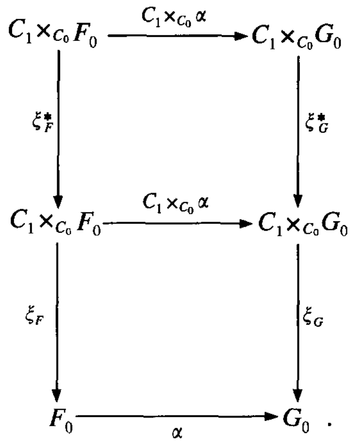
$$\begin{array}{ccc} C_1 \times_{C_0} F_0 & \xrightarrow{C_1 \times_{C_0} \alpha} & C_1 \times_{C_0} G_0 \\ {\scriptstyle \xi_F}\downarrow & & \downarrow{\scriptstyle \xi_G} \\ F_0 & \xrightarrow[\alpha]{} & G_0 \end{array}$$

is a pullback.

Proof. Point (i) is a simple checking; only observe that when $\mathbb{C}$ is Set, then ξ_F^* is the map $\xi_F^*(f,x) = \langle f^{-1}, F(f)(x)\rangle$, so that

$$\xi_F\,\xi_F^*(f,x) = \xi_F\langle f^{-1}, F(f)(x)\rangle = F(f^{-1})F(f)(x) = x.$$

As for point (ii), consider the commutative diagram

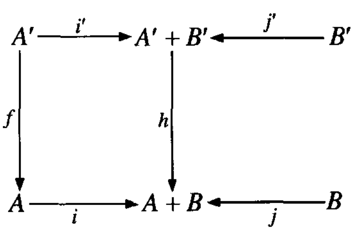


Since the external square is a pullback by point (i), it is enough to observe that the maps ξ^* are isomorphisms, since $(\xi^*)^2 = 1$:

$$\xi^*\,\xi^* = \langle \pi_1, \pi_2\rangle \xi^*\,\xi^* = \langle (-)^{-1}(-)^{-1}\pi_1, \pi_2\rangle = \langle \pi_1, \pi_2\rangle = 1. \qquad \square$$

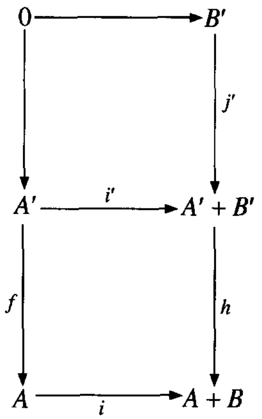
The following lemma is a basic fact about extensive categories, which follows from uniqueness of complements. We give a simple direct proof of it.

Lemma 2. *Let $\mathbb{C}$ be an extensive category and let*

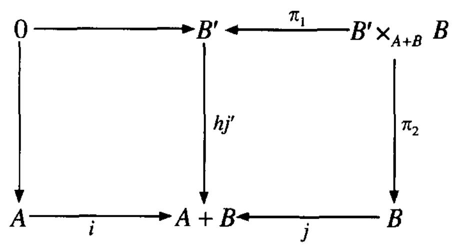


be a diagram in $\mathbb{C}$ in which the rows are coproduct diagrams and the square is a pullback. Then there exists a unique $g : B' \longrightarrow B$ such that $h\,j' = j\,g$.

Proof. Both squares in the diagram



are pullbacks, so that the whole diagram is a pullback. Therefore the top row in the diagram



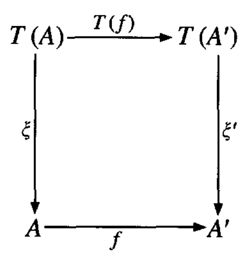
is a coproduct diagram, and hence π_1 is an isomorphism; so, we can take $g = \pi_2 \pi_1^{-1}$. Uniqueness follows from the fact that coproduct injections are monomorphisms. □

An immediate consequence of this lemma is the following useful fact:

Lemma 3. *Let $T = (T, \eta, \mu)$ be a monad on a lextensive category $\mathbb{C}$ satisfying the following conditions:*

(a) *The functor $T : \mathbb{C} \longrightarrow \mathbb{C}$ preserves finite coproducts.*

(b) *For each T-algebra morphism $(A, \xi) \xrightarrow{f} (A', \xi')$ the square*



is a pullback.

Then the forgetful functor $\mathbb{C}^T \longrightarrow \mathbb{C}$ from the category of T-algebras reflects complements.

Proof. We have to show that if

$$A \xrightarrow{f} A+B \xleftarrow{g} B$$

is a coproduct diagram in $\mathbb{C}$ such that A and $A+B$ have algebraic structures ξ_A and ξ_{A+B} respectively, making f a T-algebra homomorphism, then there exists a unique T-algebra structure ξ_B on B such that g is a T-algebra homomorphism, and

$$(A,\xi_A) \xrightarrow{f} (A+B,\xi_{A+B}) \xleftarrow{g} (B,\xi_B)$$

is a coproduct diagram in $\mathbb{C}^T$. By the previous lemma there exists a unique $\xi_B : T(B) \longrightarrow B$ such that the diagram

$$\begin{array}{ccc} T(B) & \xrightarrow{T(g)} & T(A+B) \\ \downarrow{\scriptstyle \xi_B} & & \downarrow{\scriptstyle \xi_{A+B}} \\ B & \xrightarrow[g]{} & A+B \end{array}$$

commutes and, since g is a monomorphism, then (B,ξ_B) is a T-subalgebra of $(A+B,\xi_{A+B})$. Finally, since T preserves finite coproducts, then

$$(A,\xi_A) \xrightarrow{f} (A+B,\xi_{A+B}) \xleftarrow{g} (B,\xi_B)$$

is a coproduct diagram in $\mathbb{C}^T$. □

This lemma can be applied to the monad T on $(\mathbb{C}\downarrow C_0)$ whose category of algebras is the category $\mathbb{C}^C$ of C-actions; in this case the functor

$$T : (\mathbb{C}\downarrow C_0) \longrightarrow (\mathbb{C}\downarrow C_0)$$

is the functor sending each map $\phi : A \longrightarrow C_0$ to the map $c\,\pi_1$, the map π_1 being defined by the pullback

$$\begin{array}{ccc} C_1 \times_{C_0} A & \xrightarrow{\pi_2} & A \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \phi} \\ C_1 & \xrightarrow[d]{} & C_0 \end{array},$$

and hence T satisfies condition (a) of the lemma because $\mathbb{C}$ is lextensive, and condition (b) by Lemma 1(ii). Therefore we have proved:

Theorem 4. *Let $\mathbb{C}$ be a lextensive category and let C be an internal groupoid in $\mathbb{C}$. Then the forgetful functor*

$$\mathbb{C}^C \longrightarrow (\mathbb{C}\downarrow C_0)$$

reflects complements.

A well-known instance of the theorem is when $\mathbb{C}$ is Set and C is a group: if S is a set on which the group C acts and S' is a subset of S on which the action of C restricts, then the action of C restricts also to the complement $S \setminus S'$, and $S = S' + (S \setminus S')$ in the category of C-sets. Recalling that a presheaf topos $[\mathbb{C}^{\mathrm{op}}, \mathrm{Set}]$ is *boolean* when $\mathbb{C}$ is a groupoid, then Theorem 4 should be seen as a generalization of this fact.

Another instance of the theorem is when $E \xrightarrow{p} B$ is an *effective descent morphism* in $\mathbb{C}$, which means that the pullback functor

$$(\mathbb{C}\downarrow B) \xrightarrow{p^*} (\mathbb{C}\downarrow E)$$

is monadic. Then $(\mathbb{C}\downarrow B)$ is canonically equivalent to $\mathbb{C}^{Eq(p)}$, where $Eq(p)$ is the internal groupoid given by the kernel pair of p as an equivalence relation. When $\mathbb{C}$ is a topos, the p is an effective descent morphism if and only if it is an *epimorphism*, and the fact that in this case p^* reflects complements is well-known.

3. Decidable (= separable) objects and morphisms

Definition 5. Let $\mathbb{C}$ be a lextensive category.

(a) An object D of $\mathbb{C}$ is said to be 'decidable' (or 'separable') if its diagonal has a complement, i.e. if there exists a map $\tilde{D} \xrightarrow{\varepsilon_D} D \times D$ such that

$$D \xrightarrow{\delta_D} D \times D \xleftarrow{\varepsilon_D} \tilde{D}$$

is a coproduct diagram in $\mathbb{C}$;

(b) A map $A \xrightarrow{f} B$ is decidable when is decidable as an object of the lextensive category $(\mathbb{C}\downarrow B)$.

There are two classical cases in which the notion of decidable object has been extensively studied:

Theorem 6 (Johnstone and Linton [13, Theorem 1.1]). *The following conditions are equivalent in a topos $\mathbb{C}$:*

(i) *the full subcategory $\mathbb{C}_{Kf}$ of $\mathbb{C}$ determined by Kuratowski-finite objects is a topos;*

(ii) $\mathbb{C}_{Kf}$ *is a boolean topos*;
(iii) *every Kuratowski-finite object is decidable.*

Theorem 7 (Acuña-Ortega and Linton [1, Lemma 1.3]). *The following conditions are equivalent for an object D in a topos $\mathbb{C}$ with natural numbers object*:
(i) *D is Kuratowski-finite and decidable*;
(ii) *there exists an object V such that $V \longrightarrow 1$ is an epimorphism and the projection $V \times D \longrightarrow V$ is a finite cardinal in $(\mathbb{C} \downarrow V)$.*

The following remark is also of interest:

Remark 8 (Acuña-Ortega and Linton [1, Remark 2.6]). A topos $\mathbb{C}$ is boolean if and only if every object of $\mathbb{C}$ is decidable.

The other crucial example is in the lextensive category given by the dual category of commutative rings:

Example 9. Decidable maps $A \xrightarrow{f} B$ in the lextensive category

$$\mathbb{C} = (\text{Commutative Rings})^{\text{op}}$$

are precisely separable B-algebras A.

Let us now consider the example of the lextensive category Top of topological spaces, which, as in the previous example, does not have almost any exactness property besides lextensivity.

Example 10. The decidable objects in Top are clearly just discrete spaces, but a map $A \xrightarrow{f} B$ is decidable if and only if both the following conditions hold:
(a) The diagonal Δ_A is open in the kernel of f, which means that f is 'locally injective': for each point $a \in A$ there exists an open neighborhood U of a such that the restriction of f to U is injective.
(b) The complement of the diagonal Δ_A is open in the kernel of f, which means that A is an 'Hausdorff space over B': for each pair of points $a, a' \in A$ with $a \neq a'$ and $f(a) = f(a')$, there exist two open neighborhoods U of a and U' of a' which are disjoint.
So, if A is a Hausdorff space, then f is decidable if and only if it is locally injective.

Our purpose is to give a complete list of the 'standard properties' of decidable objects and maps which hold in *any* lextensive category. Of course they are mostly known in the case of topos and as properties of separable algebras.

Theorem 11. *Let $\mathbb{C}$ be a lextensive category. Then*:
(1) *Any subobject of a decidable object is decidable.* 0 *and* 1 *are decidable.*
(2) *All monomorphisms are decidable maps*; *moreover, all composites of a monomorphism followed by a decidable map are decidable maps.*
(3) *$A + B$ is decidable if and only if both A and B are decidable.*

(4) *A map*

$$\Theta_X = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : A_1 + A_2 \longrightarrow B$$

is decidable if and only if both f_1 *and* f_2 *are decidable.*

(5) *Let* $A_i \xrightarrow{f_i} B_i$, $i = 1,2$, *be maps in* $\mathbb{C}$; *then* $f_1 + f_2$ *is decidable if and only if both* f_1 *and* f_2 *are decidable.*

(6) *If*

$$X \overset{e}{\hookrightarrow} Y \underset{g}{\overset{f}{\rightrightarrows}} Z$$

is an equalizer diagram and Z *is decidable, then* X *is a complemented subobject of* Y *and, conversely, if for each parallel pair* f, g *of morphisms with codomain* Z *the equalizer is complemented, then* Z *is decidable.*

(7) *If* $A \xrightarrow{f} B$ *is a map in* $\mathbb{C}$ *such that* B *is decidable, then the kernel pair of* f *as a subobject of* $A \times A$ *is complemented.*

(8) *If* B *is a decidable object and* $A \xrightarrow{f} B$ *is a decidable map, then* A *is decidable.*

(9) *Decidable maps are closed under composition.*

(10) *Every map with decidable domain is decidable.*

(11) *If a composite is decidable, then the first map is decidable.*

(12) *Pulling back along any map* $E \xrightarrow{p} B$ *of* $\mathbb{C}$ *preserves decidability and, when* p *is an effective descent morphism, then reflects decidability.*

(13) *If* D *is a decidable object, then for all* V *the projection*

$$V \times D \longrightarrow V$$

is a decidable map and, if the map $V \longrightarrow 1$ *is an effective descent morphism, then the converse holds too.*

(14) *Let* $A_i \xrightarrow{f_i} B_i$, $i = 1,2$, *be maps in* $\mathbb{C}$ *if* f_i *are decidable, then* $f_1 \times f_2$ *is decidable; if* $f_1 \times f_2$ *is decidable and* $A_1 \longrightarrow 1$ *is an effective descent morphism, then* f_2 *is decidable.*

(15) *Finite products of decidable objects are decidable, and if* $A \times B$ *is decidable with* $B \longrightarrow 1$ *an effective descent morphism, then* A *is decidable.*

(16) *Let* $A_i \xrightarrow{f_i} B_i$, $i = 1,2$, *be maps in* $\mathbb{C}$ *and let* $f = A_1 \times_B A_2 \longrightarrow B$ *be the map* $f_1 \pi_1 = f_2 \pi_2$; *if* f_i *are decidable, then* f *is decidable; if* f *is decidable and* f_1 *is an effective descent morphism, then* f_2 *is decidable.*

(17) *Let*

$$\begin{array}{ccccc} A_1 & \xrightarrow{g} & C & \longleftarrow & A_2 \\ {\scriptstyle f_1}\downarrow & & \| & & \downarrow{\scriptstyle f_2} \\ B_1 & \longrightarrow & C & \longleftarrow & B_2 \end{array}$$

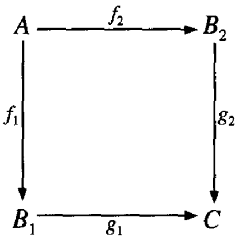
be a commutative diagram in $\mathbb{C}$; *if* f_i *are decidable, then*

$$f_1 \times_C f_2 : A_1 \times_C A_2 \longrightarrow B_1 \times_C B_2$$

is decidable and, if $f_1 \times_C f_2$ *is decidable and* g *is an effective descent morphism, then* f_2 *is decidable.*

(18) *Let* $A \xrightarrow{f_i} B_i$, $i = 1,2$, *be maps in* $\mathbb{C}$; *if* f_1 *or* f_2 *is decidable, then* $\langle f_1, f_2 \rangle : A \longrightarrow B_1 \times B_2$ *is decidable and, if* $\langle f_1, f_2 \rangle$ *and* B_1 *are decidable, then* f_2 *is decidable.*

(19) *Let*



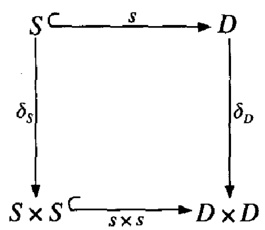
be a commutative diagram in $\mathbb{C}$; *if* f_1 *or* f_2 *is decidable, then* $\langle f_1, f_2 \rangle : A \longrightarrow B_1 \times_C B_2$ *is decidable and, if* $\langle f_1, f_2 \rangle$ *and* g_1 *are decidable, then* f_2 *is decidable.*

(20) *Let* T *be a monad satisfying the conditions of Lemma* 3 *and* (A, ξ) *be a* T*-algebra; then* (A, ξ) *is decidable in the lextensive category* $\mathbb{C}^T$ *of* T*-algebras if and only if so is* A. *In particular, if* C *is an internal groupoid in* $\mathbb{C}$ *and* F *is an internal* C*-action, then* F *is decidable in the lextensive category* $\mathbb{C}^C$ *of* C*-actions if and only if so is* F_0.

Proof. (1) If

$$S \overset{s}{\hookrightarrow} D$$

is a subobject of a decidable object D, then the diagram



is a pullback, so that by lextensivity

$$S \xrightarrow{\delta_S} S \times S \longleftarrow (S \times S) \times_{D \times D} \tilde{D}$$

is a coproduct diagram.

(2) Follows immediately from (1).

(3) Since injections are monic, the part 'only if' follows from (1). If A and B are decidable, then the diagram

$$A+B \xrightarrow{\delta_A+\delta_B} (A\times A)+(B\times B) \xleftarrow{\varepsilon_A+\varepsilon_B} \tilde{A}+\tilde{B}$$

is a coproduct diagram; on the other hand, extensivity gives that $(A\times A)+(B\times B)$ is a complemented subobject of $(A+B)\times(A+B)$, so that it will be enough to show that the composition

$$A+B \xrightarrow{\delta_A+\delta_B} (A\times A)+(B\times B) \longrightarrow (A+B)\times(A+B),$$

where the second map is $\langle \pi_1^A+\pi_1^B, \pi_2^A+\pi_2^B\rangle$ (the π's denoting the appropriate projections), is the diagonal δ_{A+B}, which is an easy calculation.

(4) Follows from (3) applied to the lextensive category $(\mathbb{C}\downarrow B)$.

(5) Follows directly from the extensivity axiom: the functor

$$(\mathbb{C}\downarrow B_1)\times(\mathbb{C}\downarrow B_2) \xrightarrow{+} (\mathbb{C}\downarrow(B_1+B_2))$$

is an equivalence.

(6) Simply describe the equalizer as the pullback of the diagonal of Z along $\langle f,g\rangle$; as for the converse, use projections.

(7) Just describe the kernel pair as an equalizer.

(8) The composition $A \longrightarrow A\times_B A \longrightarrow A\times A$ is the diagonal of A, and since A is complemented in $A\times_B A$ and $A\times_B A$ is complemented in $A\times A$ by (7), we conclude that A is complemented in $A\times A$.

(9) Obvious, from (8) applied to comma categories.

(10) Follows simply by observing that the square

$$\begin{array}{ccc} A & \longrightarrow & A\times_B A \\ \| & & \downarrow \\ A & \longrightarrow & A\times A \end{array}$$

is a pullback.

(11) Follows from (10) applied to the comma category. Notice that the decidability of a composite and of the first map does not imply the decidability of the second, even in a topos.

(12) Follows from the fact that the functor inverse image p^* preserves pullbacks and finite coproducts; when p is an effective descent morphism, then the converse follows from the fact that p^*, which is now monadic, reflects pullbacks and by Lemma 3 also reflects complements, as we already mentioned at the end of Section 2.

(13) Obvious from (12) applied to $p = (V \longrightarrow 1)$.

(14) Follows from (9), applied to the map

$$f_1 \times f_2 \longrightarrow f_1 \longrightarrow 1,$$

in the arrow category of $\mathbb{C}$, which is lextensive when $\mathbb{C}$ is, because $f_1 \times f_2 \longrightarrow f_1$ is decidable by point (13). As for the converse, since $f_1 \times f_2 = (f_1 \times 1)(1 \times f_2)$, then $(1 \times f_2)$ is decidable, by point (11); then consider the diagram

$$\begin{array}{ccc} A_1 \times A_2 & \xrightarrow{\pi_2} & A_2 \\ {\scriptstyle 1 \times f_2}\downarrow & & \downarrow{\scriptstyle f_2} \\ A_1 \times B_2 & \xrightarrow[\pi_2]{} & B_2 \end{array} \;;$$

since the diagram is a pullback, we can apply point (12), so that we only need to know that the projection $A_1 \times B_2 \longrightarrow B_2$ is an effective descent morphism, which follows from the fact that

$$\begin{array}{ccc} A_1 \times B_2 & \xrightarrow{\pi_2} & B_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow \\ A_1 & \longrightarrow & 1 \end{array}$$

is a pullback [1]

(15) Follows from (14).

(16) Follows from (15) applied to the comma category.

(17) Follows from (14) applied to the comma category.

(18) Follows from the fact that $f_i = \pi_i \langle f_1, f_2 \rangle$ $(i = 1,2)$, and from point (11); the converse when B_1 is decidable follows from the fact that the projection $B_1 \times B_2 \longrightarrow B_2$ is a decidable map by point (13), so that we can apply point (9) to $f_2 = \pi_2 \langle f_1 . f_2 \rangle$.

(19) Follows from (18) applied to the comma category.

(20) Follows from Lemma 3. □

Corollary 12. *If* $\mathbb{C}$ *is a lextensive category, then the full subcategory* Dec($\mathbb{C}$) *of* $\mathbb{C}$ *determined by decidable objects is lextensive and is full on subobjects.*

[1] For the stability under pullbacks of effective descent morphisms, see [17, Theorem 3.1].

Remark 13. As mentioned in the introduction, most of these properties are known in two cases, namely when $\mathbb{C}$ is either a topos or the dual of commutative rings. However, we would like to point out the following:

1. Many of the properties of decidable objects in a topos are established by Acugña Ortega and Linton in [1], and by Johnstone in [12]. For example, properties 4.1, 4.2 and the first part of 4.3 of [1] are included in our Theorem 11 as properties (1), (3), (4) and (6) respectively; properties 3.1(i) of [12] is the same as property (8) in our Theorem 11 (in the case of toposes). Also, some of our proofs are just the same of [1, 12]. However, let us compare the observations 2.4 and 2.5 of [1] with point (12) of our Theorem 11 and with our observation that p^* reflects complements, which we made at the end of Section 2. In the case of a topos, the requirement on p (p must be just an epimorphism) and the proof are almost trivial: if p is an epimorphism, then p^* reflects isomorphisms and, since p induces a logical functor, then it also reflects complements. In the case of a lextensive category there are at least four different reasonable notions to consider instead of that of an epimorphism, which are all equivalent in toposes:

- epimorphisms,
- regular epimorphisms,
- pullback-stable regular epimorphisms (= descent morphisms, if coequalizers of equivalence relations exist),
- effective descent morphisms,

and it seems that the best thing we can do is to take p to be an effective descent morphism, and use lextensivity and monadicity as in Section 2.

2. Consider the basic properties of commutative separable algebras over commutative rings, as presented e.g. in the book of DeMeyer and Ingraham (see Section 1 of Ch. 2 of [7]). The commutative version of all of them (1.6 to 1.13) follows from our Theorem 11. Yet, the proofs are different and, *again*, particularly different is the proof of the reflection of decidability = separability by the pullback functor (which now is the tensor product functor). The role of an effective descent morphism $p:E \longrightarrow B$ is played by a ring homomorphism, say $p:R \longrightarrow S$, such that R is a direct summand of S as an R-module (recall that we are working in the dual category of the category of commutative rings, so that R plays the role of B and S the role of E). Accordingly, all proofs are module-theoretic.

3. From point (6) of Theorem 11, it follows that *any retraction $X \longrightarrow Y$ with decidable Y is complemented.* This fact is well-known and very important, both in topos theory and in Galois theory of commutative rings.

In topos theory it shows that if the subobject classifier Ω is decidable, then the generic subobject true$:\mathrm{t} \longrightarrow \Omega$ is complemented, that is the topos is boolean.

In Galois theory of commutative rings it tells us that if T is a commutative separable R-algebra and if $h:T \longrightarrow R$ is an R-algebra homomorphism, then there exists a (unique) idempotent $e \in T$ with $h(t)e = te$, for all $t \in T$, and $h(e) = 1$ – a result whose usefulness is well-known (see e.g. [7, p. 86, Lemma 1.5]).

4. Finite coverings in lextensive categories

Let $\mathbb{C}$ be a lextensive category and, recalling that the category $\mathrm{Set}_{\mathrm{fin}}$ of finite sets is 2-initial in the 2-category of lextensive categories, for each object C consider the canonical functor

$$\mathrm{Set}_{\mathrm{fin}} \xrightarrow[\Delta_C]{} (\mathbb{C} \downarrow C).$$

Definition 14. A map $\alpha : A \longrightarrow B$ in $\mathbb{C}$ is said to be a 'finite covering' if there exists an effective descent morphism $p : E \longrightarrow B$ with connected E and a finite set n such that $p^*(\alpha) \simeq \Delta_E(n)$. The map p is then called a 'trivializing cover'.

If $\mathbb{C}$ is a topos, this definition is well-known (see [2]); it comes from Topology and is usually considered in the case where B is connected (note that our definition of a finite covering makes sense only for connected B). In the case of $\mathbb{C}$ the dual of commutative rings is also well-known: if B is a field, then $\alpha : A \longrightarrow B$ is a finite cover if and only if A is a separable B-algebra; the same is true for any connected commutative ring B, but 'separable' must be replaced by 'strongly separable'. This is briefly mentioned in [9], and the corresponding fact for étale coverings in Algebraic Geometry is known since Grothendieck [8]. In the 'known cases' above, every finite covering is a decidable morphism, and in fact this is true in any lextensive category:

Theorem 15. *If $\alpha : A \longrightarrow B$ is a finite covering in $\mathbb{C}$, then α is a decidable morphism.*

Proof. Let $p : E \longrightarrow B$ be a trivializing cover; then, since the identity of E is a decidable morphism, $\Delta_E(n)$ is a decidable object in $(\mathbb{C} \downarrow E)$ by Theorem 11, (4). Therefore $p^*(\alpha)$ is a decidable object in $(\mathbb{C} \downarrow E)$ and hence α is a decidable morphism, by Theorem (11, 12). □

Remark 16. Galois theory in categories (see [9, 11]) provides a general notion of a covering with respect to an adjunction

$$\mathbb{C} \underset{H}{\overset{I}{\rightleftarrows}} \mathbb{X} \quad (I \dashv H)$$

(the connection with coverings in toposes is explained in [10]). In such a situation, when $\mathbb{X}$ and $\mathbb{C}$ are lextensive, Theorem 15 can be generalized as follows: if

(i) every morphism of $\mathbb{X}$ is decidable,

(ii) H preserves binary coproducts,

then all coverings with respect to the adjunction are decidable. Moreover, under conditions (i) and (ii), every morphism $\alpha : A \longrightarrow B$ in $\mathbb{C}$ for which the induced morphism $A \longrightarrow B \times HIA$ is a monomorphism, is decidable (such morphisms were considered in [4], and called '$\mathbb{X}$-discrete' morphisms).

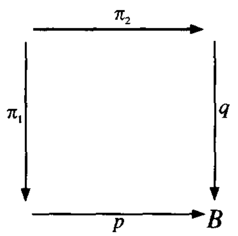
Remark 17. Theorem 15 is a basic ingredient in order to develop the Galois theory of decidable objects in a lextensive category. Let us point out that the abstract approach to Galois theory does not use any underlying 'linear' category. In particular, there are no commutative rings: the lextensive approach only remembers that the dual of the category of commutative rings is a lextensive category, and that there is a basic adjunction between the category of commutative rings and the category of profinite spaces.

A quite different approach is given in [3], where an abstract notion of a '(strongly) separable algebra' in a given 'linear category $\mathcal{V}$' (= symmetric monoidal closed and additive category) is discussed. The main purpose there was to find under which conditions on the underlying linear category $\mathcal{V}$, the dual category of the category of separable algebras in $\mathcal{V}$ has 'good quotients', i.e. it is a pretopos and, in particular, when it is a Galois category in the sense of Grothendieck; such a discussion led to a *characterization* of categories of continuous linear representations of profinite groups as *abstract categories.* A somewhat related question in the context of toposes is discussed in [12].

We now *characterize* decidable morphisms which are *finite coverings*, and let us start with some general definitions.

Definition 18. Let $\mathbb{C}$ be a lextensive category and let $\mathbb{D}$ be a class of morphisms with given codomain B, i.e. a class of objects of $(\mathbb{C}\downarrow B)$; we will say that $\mathbb{D}$ is a 'proper class' if the following conditions hold:

(i) $\mathbb{D}$ is closed under finite products in $(\mathbb{C}\downarrow B)$, that is $\mathbb{D}$ contains the identity map 1_B, and if



is a pullback diagram with p and q in $\mathbb{D}$, then $p\pi_1 = q\,p_2$ is in $\mathbb{D}$.

(ii) if

$$X \xrightarrow{\;i\;} A \xleftarrow{\;j\;} Y$$

is a coproduct diagram in $\mathbb{C}$, then for each morphism $\alpha : A \longrightarrow B$ in $\mathbb{D}$, the composite $\alpha i : X \longrightarrow B$ is in $\mathbb{D}$ (and hence also αj is in $\mathbb{D}$).

There are many examples of proper classes; some of them are the following:

Example 19. The class of all morphisms of codomain B is proper.

Example 20. Let $\alpha : A \longrightarrow B$ be a morphism with connected codomain B. The class of all maps $p : E \longrightarrow B$ such that there exist a natural number n and a morphism $q : F \longrightarrow B$ with

$$p + q \simeq \alpha^n$$

in $(\mathbb{C} \downarrow B)$, is proper.

Example 21. The class of all decidable morphisms with codomain B is proper.

Example 22. The class of all finite coverings of (connected) B is proper.

Example 23. In the lextensive category of topological spaces, the class of all local homeomorphisms with codomain a given space B is proper.

Definition 24. Let

$$\alpha : A \longrightarrow B$$

be a morphism in $\mathbb{C}$ with connected codomain B; we will say that (i) α is '*componentially surjective*' if, for each coproduct injection

$$i : X \longrightarrow A$$

with connected X, the composition $\alpha i : X \longrightarrow B$ is an effective descent morphism;

(ii) α is '$\mathbb{D}$-*surjective*' for a proper class $\mathbb{D}$ of morphisms of codomain B, if any pullback of α along any morphism of $\mathbb{D}$ is componentially surjective;

(iii) α has a '*finite* $\mathbb{D}$-*rank*' for a proper class $\mathbb{D}$ of morphisms of codomain B, if there exists a natural number n such that for any morphism

$$E \longrightarrow B$$

in $\mathbb{D}$ with connected domain E there exists a decomposition

$$E \times_B A \simeq E_1 + \cdots + E_m$$

with connected $E_1, \ldots, E_m$ and $m \leq n$.

Theorem 25. *Let $\mathbb{C}$ be a lextensive category. The following are equivalent for a decidable morphism $\alpha : A \longrightarrow B$ with connected codomain B:*

(i) *there exists a proper class $\mathbb{D}$ of morphisms with codomain B such that*

(1) *$\mathbb{D}$ contains α,*

(2) *α is $\mathbb{D}$-surjective,*

(3) *α has a finite $\mathbb{D}$-rank.*

(ii) *α is a finite covering.*

Proof. (i)$\Rightarrow$(ii): Choose an effective descent morphism $p:E \longrightarrow B$ in $\mathbb{D}$ such that $E \times_B A$ has a maximal number of connected components – say n:

$$E \times_B A \simeq E_1 + \cdots + E_n .$$

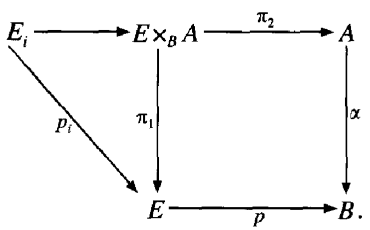
Hence, in $(\mathbb{C} \downarrow E)$ we have

$$E \times_B A \xrightarrow{\pi_1} E \simeq p_1 + \cdots + p_n$$

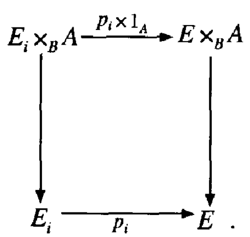
where the p_i $(i = 1, \ldots, n)$ are the composites

$$E_i \longrightarrow E_1 + \cdots + E_n \simeq E \times_B A \xrightarrow{\pi_1} E .$$

From the properties of decidable morphisms we have proved (points (2) and (12) of Theorem 11), we know that each p_i is decidable. We also know that each p_i is an effective descent morphism, because α is $\mathbb{D}$-surjective; moreover, the composition $p\,p_i : E_i \longrightarrow B$ is in $\mathbb{D}$, as we can see from the diagram



Consider now the pullback



Since p_i is an effective descent morphism, each component of $E \times_B A$ gives a (different) non-zero component of $E_i \times_B A$, and since n is maximal and $p\,p_i$ is in $\mathbb{D}$ and is an effective descent morphism,[2] each connected component of $E \times_B A$ gives a connected component of $E_i \times_B A$. In particular, $E_i \times_E E_i$ is connected. Since $p_i : E_i \longrightarrow E$ is decidable, this means that the diagonal $E_i \longrightarrow E_i \times_E E_i$ is an isomorphism, i.e. p_i is a monomorphism, and hence an isomorphism, because it is an effective descent morphism. So, $p^*(\alpha) \simeq \Delta_E(n)$, hence α is a finite covering.

(ii)$\Rightarrow$(i): Every finite covering $\alpha : A \longrightarrow B$ is decidable, by Theorem 15, and we can take as $\mathbb{D}$ the class of all morphisms with codomain B. Conditions (2) and

[2] For the closure of effective descent morphisms under composition, see e.g. [15].

(3) easily follow from the fact that pulling back along effective descent morphisms preserves and reflects non-trivial finite coproduct diagrams, and from the fact that if

$$\begin{array}{ccc} E\times_B A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ E & \xrightarrow[p]{} & B \end{array}$$

is a pullback diagram in which p is an effective descent morphism, then α is an effective descent morphism if and only if so is π_1 (see [15]). □

5. Two applications

We will now discuss two applications of the main theorem of the last section, and let us start by showing how it specializes to the Galois theory of commutative rings, and hence the classical Galois theory. Let $\mathbb{C}$ be the dual category of commutative rings

$$\mathbb{C} = (\text{Commutative Rings})^{\text{op}}.$$

As already mentioned, a *decidable morphism* $f: A \longrightarrow B$ in $\mathbb{C}$ is precisely a *separable B-algebra* (A is separable over B), and all the basic properties of commutative separable algebras follow from the properties of decidable objects in a lextensive category considered in Section 3. Therefore, it is a natural question to ask if the Galois theory of commutative rings can be generalized to an arbitrary lextensive category. In fact, since the Galois theory can be developed even in a general category with pullbacks with respect to a given adjunction, as recalled in Remark 16, and the Galois theory of commutative rings (which is a particular case) deals with the so-called 'strongly separable algebras', the question reduces to the following: can Theorem 25 provide a simple proof of the fact that strongly separable algebras are finite coverings in $\mathbb{C}$? The positive answer follows from the following

Proposition 26. *Let $\alpha: A \longrightarrow B$ be a strongly separable B-algebra A over a connected B; then α satisfies condition* (i) *of Theorem* 25, *with $\mathbb{D}$ the class of all morphisms with codomain B in $\mathbb{C}$.*

Proof. Recall that A is a strongly separable B-algebra when it is a separable B-algebra which is a finitely generated projective as a B-module. We only need to show that the morphism $\alpha: A \longrightarrow B$ in $\mathbb{C}$ satisfies condition (i) of Theorem 25, and we will show that it does with respect to the class $\mathbb{D}$ of all morphisms in $\mathbb{C}$ of codomain B. Just observe the following: let R a commutative ring and S a commutative R-algebra which is projective as on R-module; then

1. if R is connected, then S has a rank $r_R(S)$ as a projective R-module;

2. given an homomorphism $R \longrightarrow R'$, if $r_R(S)$ exists, then $r_{R'}(R' \otimes_R S)$ exists and it is the same as $r_R(S)$ (i.e. the rank is invariant under pullbacks in $\mathbb{C}$);

3. if R is connected and if $S \simeq S_1 \times S_2$ (i.e. $S \simeq S_1 + S_2$ in $\mathbb{C}$), then $r_R(S) = r_R(S_1) + r_R(S_2)$;

4. $r_R(S) = 0$ if and only if S is the zero ring;

5. if $r_R(S)$ exists, then S is faithfully flat as an R-module, in which case $S \otimes_R (-)$ (= the pullback functor in $\mathbb{C}$ along $S \longrightarrow R$) reflects isomorphisms and preserves coequalizers in $\mathbb{C}$, and hence in $\mathbb{C}$ the morphism $S \longrightarrow R$ is an effective descent morphism. □

We now come to a topological application. Let $\mathbb{C}$ be the category of topological spaces, B be a connected space, and $\alpha : A \longrightarrow B$ be a decidable continuous map. The topological meaning of decidability has been described in Example 10; in particular, when A is Hausdorff, then any local homeomorphism is decidable, since in this case decidability is equivalent to local injectivity. By applying Theorem 25 we obtain a necessary and sufficient condition for α to be a finite covering in the sense of Definition 14. Moreover, any different choice of a proper class $\mathbb{D}$ will give a somewhat different condition. We can even replace $\mathbb{C}$ itself by a 'smaller' category, say the category Étale(B) which is the full subcategory of $(\mathbb{C} \downarrow B)$ of all local homeomorphisms (in this case our Definition 14 is obviously equivalent to the classical one). For example, Theorem 25 provides a simple proof of the following

Proposition 27. *Let $\alpha : A \longrightarrow B$ be a local homeomorphism of connected Hausdorff spaces satisfying the following conditions*:

(i) *B is path connected.*

(ii) *α has the path lifting property.*

(iii) *At least one fiber of α is compact (i.e. there exists a point $b \in B$ such that $\alpha^{-1}(b)$ is compact).*

Then α is a finite covering in the classical sense.

Proof. We will just sketch the proof by observing the following:

1. We can work in Étale(B), where the effective descent morphisms are simply the surjections.

2. The path lifting property is obviously pullback stable, and any map $\alpha : X \longrightarrow Y$ with this property must be surjective, provided X is non-empty and Y is path connected (the corresponding fact in Algebra is that any homomorphism from a field to a non-zero ring is injective).

3. Using local injectivity one can show that a compact fiber of α is in fact finite, and that all other fibers must have the same number of elements. □

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